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Letter Section

Pringsheim's theorem for generalized continued fractions

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Abstract: In this paper we present a generalization to generalized continued fractions of Pringsheim's theorem on the convergence of ordinary continued fractions.

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1. Introduction

In this paper we present a theorem on the convergence of generalized continued fractions (GCF's). Recently a number of papers have been written which deal with that subject. GCF's are used in the study of simultaneous rational approximation of functions using rational functions with a common denominator, a theory which was first developed by De Bruin [1,3,4]. Convergence results were given by De Bruin [3,4] and Van der Cruyssen [8]. GCF's may also be used for the computation of certain non-dominant solutions of linear recurrence relations [6,7]. Algorithms for the calculation of GCF's are discussed in [2,7]. The theorem we are going to prove is a generalization of a well-known theorem on continued fractions which is due to Pringsheim:

Theorem (Pringsheim). *The continued fraction $K(a_n/b_n)$ converges to a finite value if*

$$|b_n| \geqslant |a_n| + 1, \quad \forall n.$$

If f_n denotes its nth approximant, then

$$|f_n| < 1, \quad \forall n.$$

(Here we use the notations of [5].) The relation between our theorem and the result of De Bruin [4] is discussed.

2. Definitions and notations

We consider the pth order linear homogeneous recurrence relation

$$y(n+p)=\sum_{i=1}^{p} a_i(n)y(n+p-i), \quad n\geqslant 0 \tag{1}$$

and we will use the following notations:
- For $i=1,\ldots,p$ $\{\alpha_i(n)\}_{n=0}^{\infty}$ is the solution of (1) with initial values

$$\alpha_i(m)=\delta_{i,m+1}, \quad m=0,1,\ldots,p-1. \tag{2}$$

- For $i=1,\ldots,p$ and $j\geqslant 1$ $\{\alpha_i^j(n)\}_{n=j}^{\infty}$ is the solution of (1) with initial values

$$\alpha_i^j(j+m)=\delta_{i,m+1}, \quad m=0,1,\ldots,p-1. \tag{3}$$

Definition 1 (de Bruin [4]). (i) The generalized continued fraction of dimension $p-1$ associated with (1) is given by the $p-1$ sequences of approximants

$$\left\{\frac{\alpha_1(n)}{\alpha_p(n)}\right\}_{n=0}^{\infty}, \quad \left\{\frac{\alpha_2(n)}{\alpha_p(n)}\right\}_{n=0}^{\infty}, \quad \ldots, \quad \left\{\frac{\alpha_{p-1}(n)}{\alpha_p(n)}\right\}_{n=0}^{\infty}.$$

(ii) The generalized continued fraction is said to converge if

$$\lim_{n\to\infty} \alpha_i(n)/\alpha_p(n)$$

exists and is finite for $i=1,2,\ldots,p-1$.

Definition 2. We say that (1) satisfies the dominance-condition (DC) if

$$\sum_{i=2}^{p} |a_i(n)|+1\leqslant |a_1(n)|, \quad \forall n\geqslant 0. \tag{4}$$

3. Some preliminary lemmas and theorems

Lemma 1. *If* (1) *satisfies the DC* (4) *then*

$$|\alpha_p(n)|\leqslant|\alpha_p(n+1)|, \quad \forall n\geqslant 0. \tag{5}$$

Proof. By induction on n. We use the notation $r(n)=|\alpha_p(n+1)|-|\alpha_p(n)|$.

(i) It is clear that $r(m)\geqslant 0$ for $m=0,1,\ldots,p-2$.

(ii) We will show that for $n\geqslant 0$:

$$r(n+m)\geqslant 0 \quad \text{for } m=0,1,\ldots,p-2 \;\Rightarrow\; r(n+p-1)\geqslant 0.$$

We have

$$\begin{aligned} r(n+p-1)&=|\alpha_p(n+p)|-|\alpha_p(n+p-1)|\\ &=\Big|\sum_{i=1}^{p} a_i(n)\alpha_p(n+p-i)\Big|-|\alpha_p(n+p-1)|\\ &\geqslant |a_1(n)|\,|\alpha_p(n+p-1)|-\sum_{i=2}^{p}|a_i(n)|\,|\alpha_p(n+p-i)|\\ &\quad-|\alpha_p(n+p-1)| \end{aligned}$$

$$
\begin{aligned}
&= (|a_1(n)|-1)|\alpha_p(n+p-1)| - \sum_{i=2}^{p} |a_i(n)||\alpha_p(n+p-i)| \\
&\overset{\mathrm{DC}}{\geqslant} \left(\sum_{i=2}^{p} |a_i(n)|\right)|\alpha_p(n+p-1)| - \sum_{i=2}^{p} |a_i(n)||\alpha_p(n+p-i)| \\
&= \sum_{i=2}^{p} \left[|a_i(n)|(|\alpha_p(n+p-1)| - |\alpha_p(n+p-i)|)\right] \\
&= \sum_{i=2}^{p} \left[|a_i(n)| \cdot \left(\sum_{k=2}^{i} r(n+p-k)\right)\right] \\
&\geqslant 0 \quad \text{by the induction hypothesis.} \qquad \square
\end{aligned}
$$

Remark. As a consequence of Lemma 1 we have

$$\alpha_p(n) \neq 0, \quad \forall n \geqslant p-1.$$

We will use this in our main theorem.

Lemma 2.

$$\alpha_i(n) = \sum_{k=1}^{i} a_{p-i+k}(k-1)\alpha_p^k(n) \quad \textit{for } i=1,\ldots,p, \quad \forall n \geqslant i, \tag{6}$$

$$\alpha_p^j(n) = \sum_{k=1}^{p} a_k(j+k-1)\alpha_p^{j+k}(n), \quad \forall j \geqslant 1, \quad \forall n \geqslant p+j. \tag{7}$$

Proof. The proof is straightforward. $\square$

Theorem 1. *If* (1) *satisfies the DC* (4) *then*

$$\sum_{m=1}^{p-1} |\alpha_m(n)| + 1 \leqslant |\alpha_p(n)|, \quad \forall n \geqslant p-1. \tag{8}$$

Proof. We will show that

$$\sum_{i=2}^{p} \sum_{m=i}^{p} |a_m(i-2)||\alpha_p^{i-1}(n)| + 1 \leqslant |\alpha_p(n)|. \tag{9}$$

The theorem then follows immediately from (6). For $n \geqslant 2p$ we have

$$
\begin{aligned}
|\alpha_p(n)| - 1 &\overset{(6)}{=} \left|\sum_{k=1}^{p} a_k(k-1)\alpha_p^k(n)\right| - 1 \\
&\geqslant |a_1(0)||\alpha_p^1(n)| - \sum_{k=2}^{p} |a_k(k-1)||\alpha_p^k(n)| - 1 \\
&\overset{(4)}{\geqslant} (|a_1(0)|-1)|\alpha_p^1(n)| + |\alpha_p^1(n)| \\
&\quad - \sum_{i=2}^{p} \left[\left(|a_1(i-1)| - 1 - \sum_{\substack{m=2 \\ m \neq i}}^{p} |a_m(i-1)|\right)|\alpha_p^i(n)|\right] - 1 \\
&= (|a_1(0)|-1)|\alpha_p^1(n)| + \sum_{i=3}^{p} \sum_{k=i}^{p} |a_k(i-2)||\alpha_p^{i-1}(n)| + A_0
\end{aligned} \tag{10}
$$

with

$$A_0 = \sum_{i=1}^{p-1} \left[|\alpha_p^i(n)| - \left(|a_1(i)|\,|\alpha_p^{i+1}(n)| - \sum_{k=2}^{p-i} |a_k(i+k-1)|\,|\alpha_p^{i+k}(n)| \right) \right]$$
$$+ |\alpha_p^p(n)| - 1.$$

If we show that $A_0 \geqslant 0$ then (9) follows immediately from (10) and the DC (4).

Since

$$\left(|a_1(p)| - 1 - \sum_{i=2}^{p} |a_i(p)| \right) |\alpha_p^{p+1}(n)| \geqslant 0$$

we have

$$A_0 \geqslant A_0 - \left(|a_1(p)| - 1 - \sum_{i=2}^{p} |a_i(p)| \right) |\alpha_p^{p+1}(n)|.$$

Rewriting this as

$$A_0 \geqslant \left[|\alpha_p^1(n)| - \left(|a_1(1)|\,|\alpha_p^2(n)| - \sum_{k=2}^{p} |a_k(k)|\,|\alpha_p^{k+1}(n)| \right) \right] + A_1$$

with

$$A_1 = \sum_{i=1}^{p-1} \left[|\alpha_p^{i+1}(n)| - \left(|a_1(i+1)|\,|\alpha_p^{i+2}(n)| - \sum_{k=2}^{p-i} |a_k(i+k)|\,|\alpha_p^{i+1+k}(n)| \right) \right]$$
$$+ |\alpha_p^{p+1}(n)| - 1,$$

we see that A_1 has exactly the same form as A_0, and furthermore $A_0 \geqslant A_1$ since the expression between square brackets is greater than or equal to zero as a consequence of (7). We now repeat the process and we get

$$A_0 \geqslant A_1 \geqslant \cdots \geqslant A_{n-2p}$$

with

$$A_{n-2p} \geqslant \sum_{i=1}^{p-1} \left[|\alpha_p^{i+n-2p}(n)| - \left(|a_1(i+n-2p)|\,|\alpha_p^{n-2p+i+1}(n)| \right. \right.$$
$$\left. \left. - \sum_{k=2}^{p-i} |a_k(i+n-2p+k-1)|\,|\alpha_p^{i+n-2p+k}(n)| \right) \right]$$
$$+ |\alpha_p^{n-p}(n)| - \left(|a_1(n-p)| - \sum_{k=2}^{p} |a_k(n-p)| \right) |\alpha_p^{n-p+1}(n)| \tag{11}$$

since $1 \leqslant |a_1(n-p)| - \sum_{k=2}^{p} |a_k(n-p)|$ and $|\alpha_p^{n-p+1}(n)| = 1$ by definition (3). Since we also have $\alpha_p^{n-p+2}(n) = \alpha_p^{n-p+3}(n) = \cdots = \alpha_p^n(n) = 0$ we may rewrite (11) as

$$A_{n-2p} \geqslant \sum_{i=1}^{p} \left[|\alpha_p^{i+n-2p}(n)| - \left(|a_1(i+n-2p)|\,|\alpha_p^{n-2p+i+1}(n)| - \sum_{k=2}^{p} |a_k(i+n-2p+k-1)|\,|\alpha_p^{i+n-2p+k}(n)| \right) \right] \geqslant 0 \quad \text{using (7).}$$

The same kind of argument may be used to prove the theorem for $p-1 \leqslant n < 2p$. □

Corollary. *The nth convergent of the GCF, i.e. the vector* $C_n = (C_n^1, C_n^2, \ldots, C_n^{p-1})$ *with* $C_n^i = \alpha_i(n+p-1)/\alpha_p(n+p-1)$ *satisfies*

$$\sum_{i=1}^{p-1} |C_n^i| < 1, \quad \forall n \geqslant 0. \tag{12}$$

Proof. From (8) using (5) we have

$$\sum_{i=1}^{p-1} \left| \frac{\alpha_i(n+p-1)}{\alpha_p(n+p-1)} \right| + \left| \frac{1}{\alpha_p(n+p-1)} \right| \leqslant 1, \quad \forall n \geqslant 0$$

with $|1/\alpha_p(n+p-1)| > 0$. □

4. Main result

Theorem 2. *If* (1) *satisfies the DC* (4) *then the GCF associated with* (1) *converges.*

Proof. We have to prove that $\lim_{n\to\infty} \alpha_i(n)/\alpha_p(n)$ exists for $i = 1, \ldots, p-1$.

Now

$$\lim_{n\to\infty} \frac{\alpha_i(n)}{\alpha_p(n)} \text{ exists} \Leftrightarrow \sum_{k=0}^{\infty} \left(\frac{\alpha_i(k+p)}{\alpha_p(k+p)} - \frac{\alpha_i(k+p-1)}{\alpha_p(k+p-1)} \right) \text{ converges.} \tag{13}$$

If we show that

$$\left| \frac{\alpha_i(k+p)}{\alpha_p(k+p)} - \frac{\alpha_i(k+p-1)}{\alpha_p(k+p-1)} \right| \leqslant \frac{|\alpha_p(k+p)| - |\alpha_p(k+p-1)|}{|\alpha_p(k+p)|\,|\alpha_p(k+p-1)|}, \quad \forall k \geqslant 0. \tag{14}$$

then

$$\sum_{k=0}^{\infty} \left| \frac{\alpha_i(k+p)}{\alpha_p(k+p)} - \frac{\alpha_i(k+p-1)}{\alpha_p(k+p-1)} \right| \leqslant \sum_{k=0}^{\infty} \left(\frac{1}{|\alpha_p(k+p-1)|} - \frac{1}{|\alpha_p(k+p)|} \right) = \lim_{k\to\infty} \left(1 - \frac{1}{|\alpha_p(k+p)|} \right).$$

Since by Lemma 1 $\{|\alpha_p(k+p)|\}$ is monotonically increasing, it follows that the series in (13) converges absolutely, and hence the GCF converges. We will now prove (14) by induction.

(1) We first show that (14) holds for $d(p), d(p+1), \ldots, d(2p-2)$, where

$$d(n) = \frac{\alpha_i(n)}{\alpha_p(n)} - \frac{\alpha_i(n-1)}{\alpha_p(n-1)} \quad \text{with } i = 1, \ldots, p-1.$$

We have

$$d(p)=\left|\frac{a_{p-i+1}(0)}{a_1(0)}\right|\leqslant\frac{|a_1(0)|-1}{|a_1(0)|}=\frac{|\alpha_p(p)|-|\alpha_p(p-1)|}{|\alpha_p(p)||\alpha_p(p-1)|}.$$

Then, assuming that (14) holds for $d(p)$, $d(p+1),\ldots,d(p+j-1)$ with $1\leqslant j\leqslant p-2$ we have for $d(p+j)$

$$\begin{aligned}|d(p+j)|&=\left|\frac{1}{\alpha_p(p+j)}\left[\sum_{k=1}^{j}a_k(j)\alpha_i(p+j-k)+a_{j+p-i+1}(j)\right.\right.\\&\qquad\left.\left.-\left(\sum_{k=1}^{j}a_k(j)\alpha_p(p+j-k)+a_{j+1}(j)\right)\frac{\alpha_i(p+j-1)}{\alpha_p(p+j-1)}\right]\right|\\&=\left|\frac{1}{\alpha_p(p+j)}\right|\left|\sum_{k=2}^{j}a_k(j)\alpha_p(p+j-k)\left(\frac{\alpha_i(p+j-k)}{\alpha_p(p+j-k)}-\frac{\alpha_i(p+j-1)}{\alpha_p(p+j-1)}\right)\right.\\&\qquad\left.+a_{j+p-i+1}(j)-a_{j+1}(j)\,\frac{\alpha_i(p+j-1)}{\alpha_p(p+j-1)}\right|.\end{aligned}$$

Using (5), (8) and the assumption above, we find

$$\begin{aligned}|d(p+j)|&\leqslant\frac{1}{|\alpha_p(p+j)||\alpha_p(p+j-1)|}\\&\quad\times\left[\sum_{k=2}^{j}|a_k(j)|\big(|\alpha_p(p+j-1)|-|\alpha_p(p+j-k)|\big)\right.\\&\qquad\left.+|a_{j+p-i+1}(j)||\alpha_p(p+j-1)|+|a_{j+1}(j)|\big(|\alpha_p(p+j-1)|-1\big)\right]\\&=\frac{1}{|\alpha_p(p+j)||\alpha_p(p+j-1)|}\\&\quad\times\left[\left(\sum_{k=2}^{j+1}|a_k(j)|+|a_{j+p-i+1}(j)|\right)|\alpha_p(p+j-1)|\right.\\&\qquad\left.-\sum_{k=2}^{j}|a_k(j)||\alpha_p(p+j-k)|-|a_{j+1}(j)||\alpha_p(p-1)|\right]\\&\overset{(4)}{\leqslant}\frac{(|a_1(j)|-1)|\alpha_p(p+j-1)|-\sum_{k=2}^{j+1}|a_k(j)||\alpha_p(p+j-k)|}{|\alpha_p(p+j)||\alpha_p(p+j-1)|}\\&\leqslant\frac{|\alpha_p(p+j)|-|\alpha_p(p+j-1)|}{|\alpha_p(p+j)||\alpha_p(p+j-1)|}\end{aligned}$$

where we define $a_{p+1}(j)=a_{p+2}(j)=\cdots=0$.

(2) If (14) holds for $d(n+1), d(n+2), \dots, d(n+p-1)$ then it also holds for $d(n+p)$. The proof is exactly as in 1, but the first line now reads

$$|d(n+p)| = \left| \frac{1}{\alpha_p(n+p)} \left[\sum_{k=1}^{p} a_k(n) \alpha_i(n+p-k) \right.\right.$$
$$\left.\left. - \frac{\alpha_i(n+p-1)}{\alpha_p(n+p-1)} \sum_{k=1}^{p} a_k(n) \alpha_p(n+p-k) \right] \right|. \quad \square$$

Remark 1. It is now very easy to see that the following result by De Bruin is a direct consequence of Theorem 2.

Theorem (de Bruin [4]). *The GCF associated with* (1) *with* $p = 3$ *converges if there exist* $a, b \in \mathbb{R}$, $a, b \geqslant 0$ *with*

$$\left| \frac{a_3(1)}{a_1(1) a_1(0)} \right| \leqslant \frac{b}{(1+a+b)^2},$$

$$\sup_{n \geqslant 2} \left| \frac{a_3(n)}{a_1(n) a_1(n-1) a_1(n-2)} \right| \leqslant \frac{b}{(1+a+b)^3},$$

$$\sup_{n \geqslant 1} \left| \frac{a_2(n)}{a_1(n) a_1(n-1)} \right| \leqslant \frac{a}{(1+a+b)^2}.$$

This theorem follows from Theorem 2 by applying an equivalence transformation to the GCF associated with (1) (for the case $p = 2$ see [5, p. 94]).

Remark 2. The corollary to Theorem 1 and Theorem 2 together constitute a generalization of Pringsheim's result for continued fractions.

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